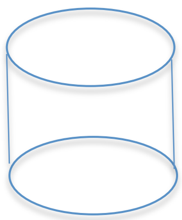
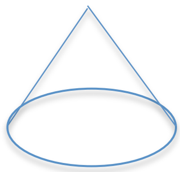
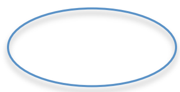
# The Limiting Case: Sublime or Ridiculous?

If you followed Chris Pritchard's 'Round peg in a square hole' series, and joined him finally [Pritchard, 9.2013] in "stretching the argument to its limit", you will already know the answer to that question: Both.

Mathematical definitions are tight but broad. If an object is defined in terms of a variable, it's worth seeing what happens at the very ends of the range of values it can take. These are the 'limiting cases'. Here are six examples from geometry. The levels of difficulty for the 5 sections are:
**1**, **6**: KS3; **2**, **5**: KS4; **3**, **4**: AS/A.

**1.** Define a (single right circular) cone as follows. Given a circle and a central, perpendicular axis, take a line segment from the circumference to the axis, making an angle $\theta$ with it. The cone is the solid of revolution of that line segment defined over the interval $0^0$ to $90^0$. Here are the possible cases:

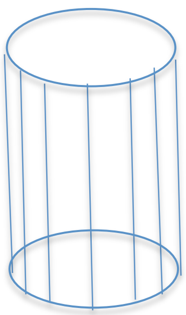
| $\theta = 0°$ | $0° < \theta < 90°$ | $\theta = 90°$ |
|---|---|---|
| special case:<br>the **cylinder** | general case:<br>the **cone** | special case:<br>the **disk** |
|  |  |  |

(You see now why I used such a clumsy definition: I didn't want the case $\theta = 0°$ to be a straight line.) Any property which does not depend on the value of $\theta$ will be shared by all cases. We may not – and generally won't – know which properties do and which do not, but we can take one and test it.

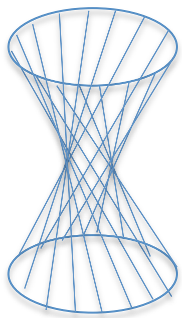
A 'developable' surface is one which can be laid out flat. The disk is already a flat surface. Are the cylinder and cone also developable? Yes: as a rectangle and circle sector respectively. The latter paper folding activity is most instructive. Follow it with this question: *What, roughly, is the semi-apical angle of the filter funnel you use in the chemistry lab?* (Hint: think of the way you fold the filter paper.) [note **1**]

**2.** Take the cone again, this time a double one, and generate the solid by rotating any line distinct from the axis. I have specified no variable but we can find the limiting cases by inspection:

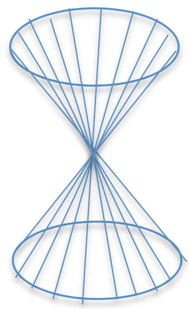
lines parallel to
the axis,
special case:
the **cylinder**



lines skew to
the axis,
general case:
the **hyperboloid**
(‘of one sheet’)Limiting



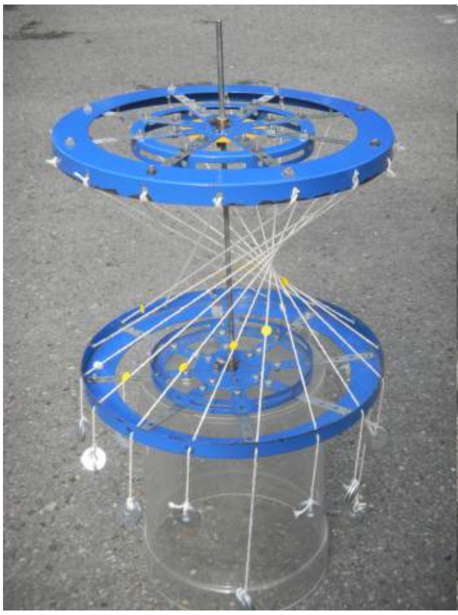
lines coincident
in a point necessarily
on the axis, special case:
the **cone**



We know that the curve produced by a plane section through the cone, and making an angle greater than the semiapical angle with the axis, is an ellipse. In the light of section **1** we can view the cylinder as a cone whose semiapical angle is zero and claim that any plane section of the cylinder making an angle greater than zero with the axis is an ellipse.

If we can obtain elliptical sections through the cone and cylinder, might the same also be possible of the hyperboloid?

A model of the ‘Cundy-&-Rollett’ type, (built as a maths club project?), allows the children to move continuously from the cylinder to the double cone. Many observations can be made beyond those relating to our immediate question [note **2**], the answer to which is ‘Yes’. Indeed *all* the conics can be obtained by slicing a hyperboloid [Mnatsakanian & Apostol, 2012]. The picture shows an ellipse in general position. (A special family shares a centre with the hyperboloid, a case not realised with the double cone.) As with the cone (but not the cylinder) the general ellipse centres lie off the rotation axis. Parallel sections of the cylinder give congruent ellipses; those of the cone, similar ones. With the hyperboloid there is no such simple relation.



What about the question we asked of example **1**: The solids at both ends of the range here are developable. What happens in between?

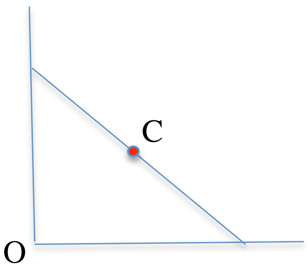
We try to wrap the hyperboloid in paper but make a right mess of it. It seems developability requires the generator to share a plane with the axis.

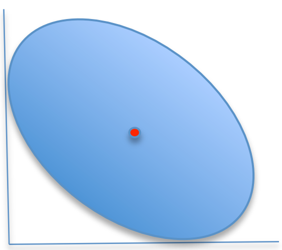
**3.** The following was a 'Putnam' problem [note **3**].

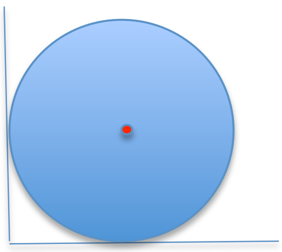
*Consider an ellipse lying in the first quadrant of the (x,y) plane and tangent to the coordinate axes. Prove that the distance from the centre of the ellipse to the origin depends only on the semiaxes of the ellipse and not on its orientation.*

If we let the length of the semi-minor axis, $b$, range between zero and the length of the semi-major axis, $a$, we have these cases:

$b = 0$ $\qquad\qquad$ $0 < b < a$ $\qquad\qquad$ $b = a$





The case $b = a$ offers no technical help but serves a heuristic purpose: it reminds us that our ellipse should make a complete revolution.

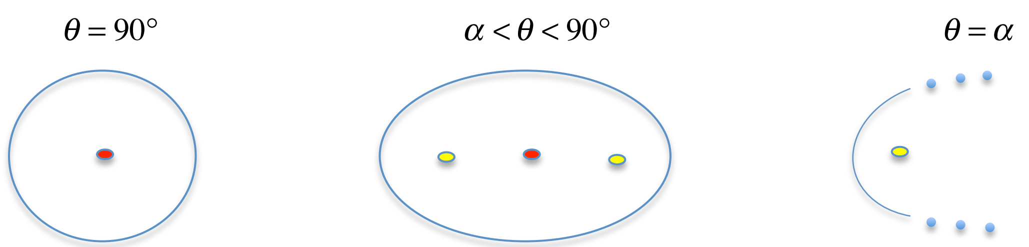
The angle subtended by a circle diameter at a point on the circumference is a right angle. Conversely, if a right angle moves so that the arms pass through the ends of a straight line, the vertex traces a circle. The case $b = 0$ invites us to swap the roles of C and O with this result. Taking their cue from this special case, our sixth form students can fix the ellipse centre at the origin and use the bookwork they have learned in their lessons on the quadratic equation and coordinate geometry in this way, [paraphrasing Bostock, Chandler & Rourke, 1982], [note **4**]:

We need the condition for a line to be tangent to an ellipse. We solve $y = mx + c$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to give a quadratic in x, then put in the condition for equal roots, obtaining $y = mx \pm \sqrt{b^2 + a^2 m^2}$ . We square this to give a quadratic in $m$, representing a pair of tangents: $(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$ , then impose the condition for the two tangents to be perpendicular, i.e. for the product of the roots to be -1, and we have $x^2 + y^2 = a^2 + b^2$ , the equation of a circle, centre the origin, radius $\sqrt{a^2 + b^2}$ .

A complete revolution of the ellipse produces a complete oscillation of the point in its circular arc.

(In the case $b = 0$ the *general* point on the line executes an ellipse, making possible the *ellipsograph* (‘trammel of Archimedes’).

**4.** Cut a cone, semi-apical angle $\alpha$ , with a plane normal to the axis, (i.e. making an angle $\theta = 90°$ with it), and rotate the plane until parallel to a generator, (thus making an angle $\theta = \alpha$ with the axis). You watch a circle become an ellipse and finally a parabola:




Centre and foci one and the same point

Centre and foci distinct

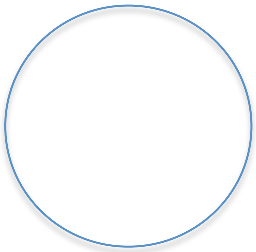
One focus remains, the centre and the other focus move off to infinity

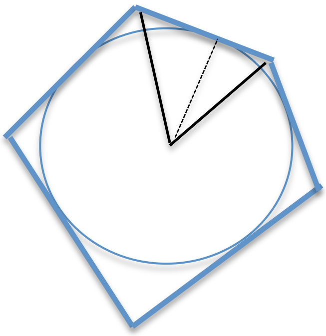
The point of that sequence is to show that, when you look along the axis of a parabola, you could be looking along the major axis of an ellipse.

Newton thought of the Moon as a projectile which didn’t land. Kepler found that the orbit of a planet round the sun – and, by extension, that of any satellite about its parent body - was an ellipse with the centre of the parent at one focus. Vi Hart [note **5**] reminds us that the path of a projectile which *does* land, though to good accuracy a parabola, is more accurately an arc of an ellipse with one focus at the centre of the Earth. A question for your A level students: *Assuming a launch angle of 45$^0$, where is the other?* [note **6**]

**5.** Mnatsakanian & Apostol [op. cit.] use the term *circumgon* for a polygon which can be circumscribed about a circle. Circumgons cover this range in terms of the number of sides, $n$:

$\infty$ | $n > 2$ | $n = 2$



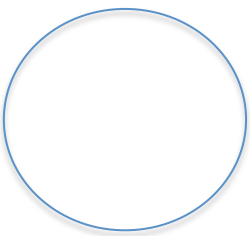


The case $n = 2$ holds no interest. But, as shown, the general circumgon can be dissected into triangles. The area of each is half the side x the altitude. Since the altitude is the radius of the incircle, we immediately have the area of the circumgon: inradius x half the circumference. Now let the number of sides approach infinity and we have the circle itself. Archimedes expressed the area of the circle in just this way when he compared it to the area of a right triangle whose legs were respectively the circumference and the radius. For a proof, Archimedes boxed the circle between regular polygons of increasing numbers of sides [note **10**]. He went further than his predecessors by estimating the ratio of half-circumference to radius, viz. $\pi$ . Your students could try the exercise nrich.maths.org/841 ('Approximating pi'). [note **7**]. The same formula emerges from the dissection of a circle we often use with our children, in which we pack sectors nose-to-tail to make a wavy parallelogram [note **8**]. We can also dissect the circle into concentric rings and build Archimedes' triangle by straightening them out [note **9**]. Your students can find all three methods at en.wikipedia.org/wiki/Area_of_a_disk , where the second and third methods are animated under the respective headings 'Rearrangement proof' and 'Triangle method'.

**6.** If you go round a polygon and sum the parts $d$ by which each vertex falls short of $\pi$ , you get $2\pi$ . If you go all over a poly*hedron* and add the parts $D$ by which the sum of the facial angles at each vertex falls short of $2\pi$ , you get $4\pi$ [note **10**]. In a regular polygon, or a regular or semiregular polyhedron, the vertices are identical so that their number, $v$, = $\dfrac{2\pi}{d}$ and $\dfrac{4\pi}{D}$ respectively.

Here are the regular polygons for $0 \le d \le \pi$ ($v \ge 2$)

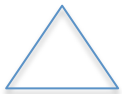
$d = 0$ ($v = \infty$) | $0 < d < \pi$ ($v > 2$) | $d = \pi$ ($v = 2$)



regular polygon

A polygon has the same number of sides and vertices so in the case $d = \pi$ the 'line' has 2 sides. I don't know what purpose this case may serve, but the case $d = 0$, seen by the Greek mathematicians of the 5th century BC as the end of a sequence of polygons with more and more sides, inspired a series of developments in integral calculus (not least the work of Archimedes, the subject of section **5**) which culminated in the limit concept 23 centuries later [note **11**].

Here are the regular and semiregular polyhedra for $0 \le D \le 2\pi$ ($v \ge 2$):

| $D = 0$ ($v = \infty$) | $0 < D < {}^{4\pi}\!/_{3}$ ($v > 3$) | $D = {}^{4\pi}\!/_{3}$ ($v = 3$) | $D = 2\pi$ ($v = 2$) |
|---|---|---|---|
| regular or semiregular tessellation | regular or semiregular polyhedron |  |  |

Like the two on the left, the two right-hand cases must satisfy Euler's polyhedron formula, $v + f = e + 2$. So, by analogy with the line in the polygon case, the 'triangle' has 2 faces. The line has an equal, but unspecified, number of edges and faces. Think of it as a globe whose surface is divided by meridia except that the slices in between (the *lunes*) have no width and zero curvature.

More interesting are the two cases on the left.

Take a particular tiling and think of it as the last member of a sequence of forms with diminishing $D$. Name a form by listing, by the number of their sides, the faces you meet as you go round a vertex. Thus a cube is 3.3.3, abbreviated $3^3$. The truncated icosahedron (football) is notated 5.6.6, abbreviated $5.6^2$. Using Polydron's 'Archimedean solids' kit (or, failing that of course, any others kits of interlocking polygons you happen to have), guide your children, (who should do all the arithmetic), along some of the following sequences. You can project or download an empty version of the chart below from www.magicmathworks.org/shapechart for completion by the children.

The sequences have no deep mathematical significance. They're just patterns. But they provide a bridge between two otherwise disparate sets of objects: polyhedra on the one hand, tilings on the other. By moving backwards and forwards along the sequences, interpolating where we find a gap, we identify candidates for further examination.

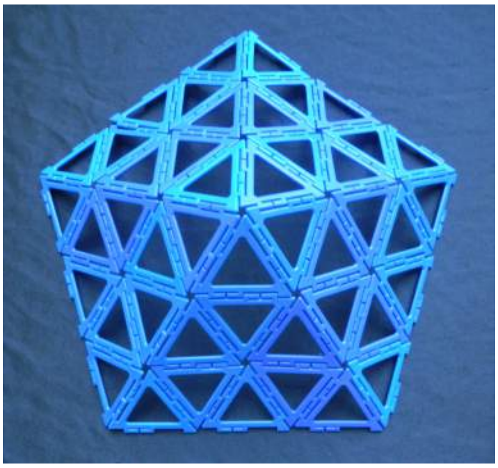
For a possible vertex, $D$ must divide $720^0$ exactly. But that does not guarantee that the vertex will repeat. For example, $5^2.6$ would be a solid with 30 vertices, but you find you can't build it. A good exercise is to tabulate all eligible forms which can be made from 3-, 4-, 5-, 6-, 8-, 10-, and 12- gons. Not counting prisms (symbol $4^2 . n$) and antiprisms (symbol $3^3 . n$), of which there are an infinite number, and counting left- and right-handed forms just once, you find there are 51, of which 29, i.e. just over half, repeat.

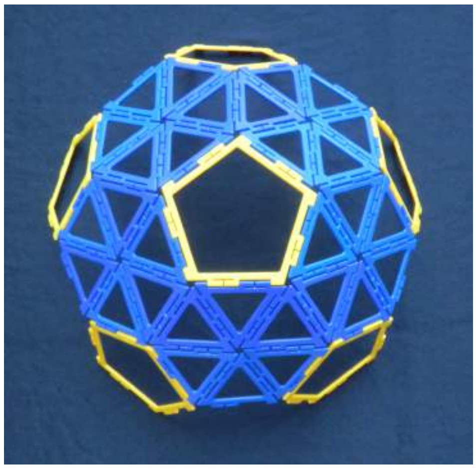
| $D$ | 180° | 120° | 90° | 60° | 36° | 30° | 24° | 15° | 12° | 6° | 0° |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 4 | 6 | 8 | 12 | 20 | 24 | 30 | 48 | 60 | 120 | $\infty$ |
| | $3^3$ | $3^4$ | | $3^5$ | | | | | | | $3^6$ |
| | | | $4^3$ | | | | | | | | $4^4$ |
| | | | | | | $3.4^3$ | | | | | $4.4^3 =$<br>$4^4$ |
| | | | | $3.6^2$ | | $4.6^2$ | | | $5.6^2$ | | $6.6^2$<br>$= 6^3$ |
| | $3^3$ | | $4^3$ | | $5^3$ | | | | | | $6^3$ |
| | | | $4^3 =$<br>$4.4^2$ | | | $4.6^2$ | | | | | $4.8^2$ |
| | | | | | | $3^4.4$ | | | $3^4.5$ | | $3^4.6$ |
| | | | | | | $3.4^3 =$<br>3.4.4.4 | | | 3.4.5.4 | | 3.4.6.4 |
| | | | | $3.6^2$ | | $3.8^2$ | | | $3.10^2$ | | $3.12^2$ |
| | | | | 3.4.3.4 | | | 3.5.3.5 | | | | 3.6.3.6 |
| | | | | $3^5 =$<br>3.3.3.3.3 | | | | | | | 3.3.4.3.4 =<br>$3^2.4.3.4$ |
| | | | | | | $3^4.4 =$<br>$3^3.3.4$ | | | | | $3^3.4.4 =$<br>$3^3.4^2$ |
| | | | | | | $4.6^2 =$<br>4.6.6 | | 4.6.8 | | 4.6.10 | 4.6.12 |

Here is the last case [note **12**]:


Of more significance are the mappings below. The arrows show operations by which you can change one form into another.

The blue arrows show *truncation*, by which you cut off corners. (We apply it here to regular forms only). You grow a new vertex for each edge you cut:





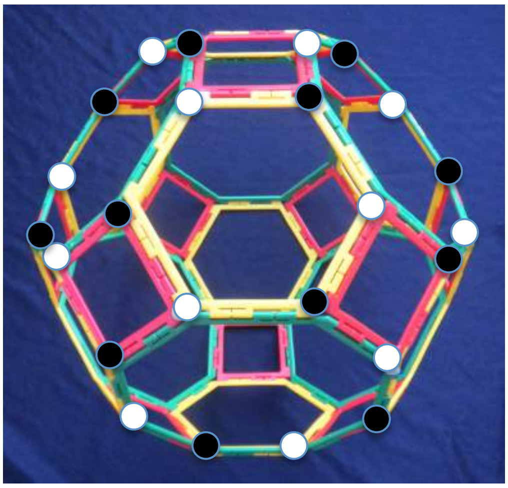
Since the number of edges meeting in a vertex is the same as the number of faces, recorded in the index, you can read it straight off the symbol. The new $v$ is then just the old $v$ times this number. Get the children to truncate one Polydron model, hazard their own rule, test it on other solids and, if successful, justify it to the class. You may then write down some of the following and the children can check the arithmetic from their charts.

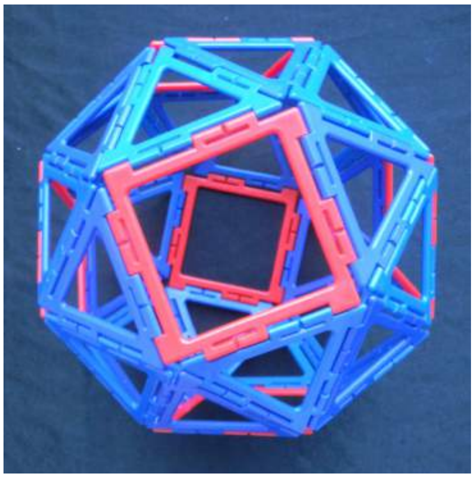
The mapping, then, shows the old index becoming the number of sides of the new face, but also the old faces doubling their number of sides:

$3^3 \xrightarrow[\text{x 2}]{} 3.6^2$

$3^3 \to 3.6^2$ $3^4 \to 4.6^2$ $3^5 \to 5.6^2$ $3^6 \to 6.6^2 = 6^3$
$4^3 \to 3.8^2$ $4^4 \to 4.8^2$
$5^3 \to 3.10^2$
$6^3 \to 3.12^2$

The green arrows show *alternation*, by which you cut off *every other* corner. It only works on forms where the faces have an even number of sides. Here we've marked with stickers the vertices we want to keep/lose. The choice of white or black determines the handedness of the result:





The effect is to halve the number of vertices throughout. You may proceed as above. In terms of the symbols, '4' disappears because a square becomes an edge. Each of the other faces contributes 2 edges to a vertex.

$4^3 = 4.4^2 \to 3^3$ $4.6^2 = 4.6.6 \to 3^4.3 = 3^5$ $4.8^2 \to 3^2.4.3.4$
$4.6.8 \to 3^4.4$
$4.6.10 \to 3^4.5$
$4.6.12 \to 3^4.6$

$4^4 \to 4^4$ (itself)

The yellow arrows show *duals*. In dual forms we swap faces for vertices, thus number and index in the symbol. In accordance with this rule, the tetrahedron and square grid are *self*-dual. (We confine ourselves to regular forms. The duals of most semiregular forms do not have regular faces.)

$3^3 \leftrightarrow 3^3$ $3^4 \leftrightarrow 4^3$ $3^5 \leftrightarrow 5^3$ $3^6 \leftrightarrow 6^3$ $4^4 \leftrightarrow 4^4$

If the children have met the Euler polyhedron formula, they will realise that the number of edges must remain unchanged. They can observe on their models the new edges set at right angles to the old.

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More examples from geometry will occur to you, and perhaps you can think of other areas of mathematics where "stretching the argument to its limit" yields results?

### *References*

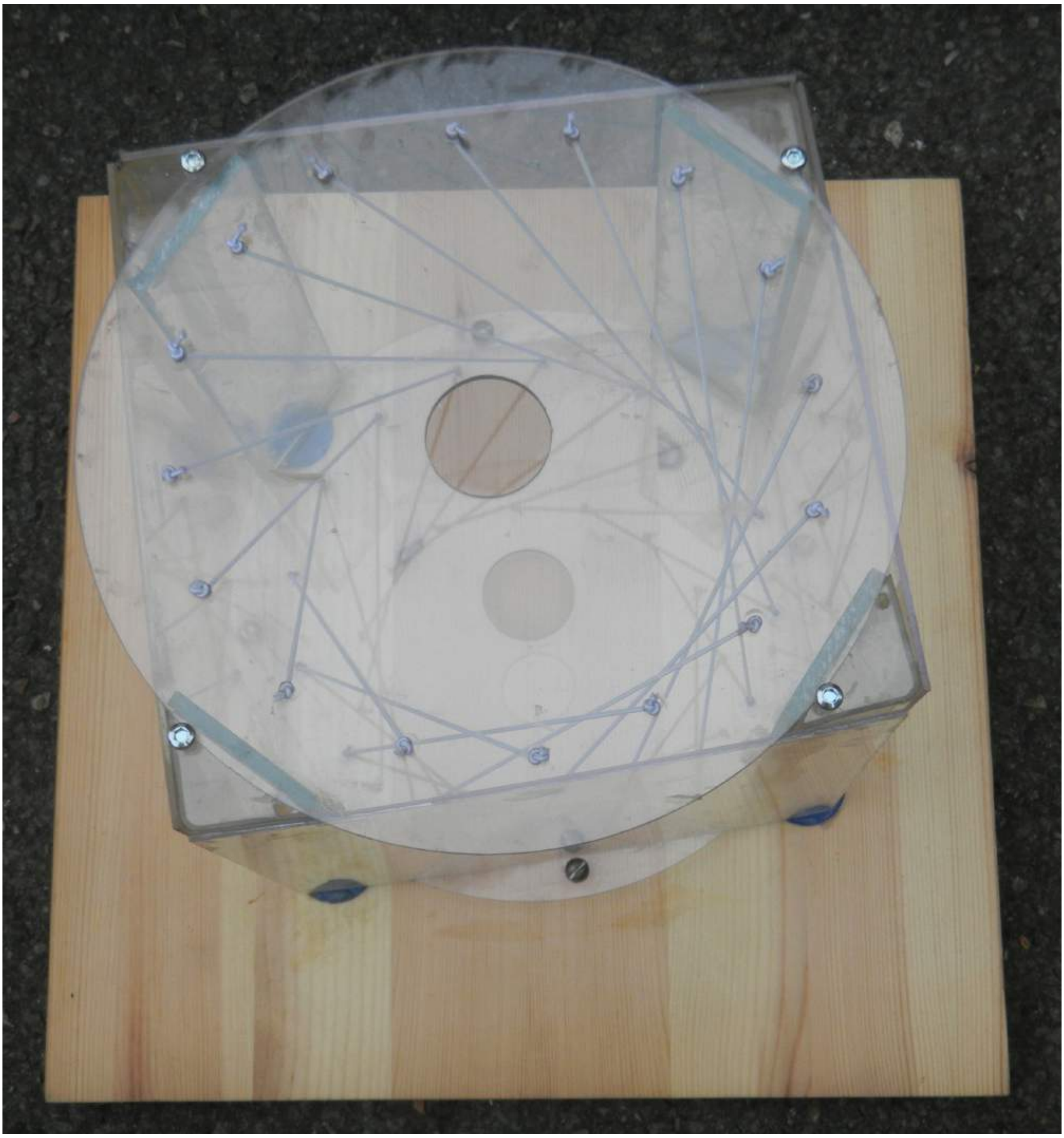
Pritchard, C. 2013 'A Square Peg in a Round Hole', *Mathematics in School,* **42**, 4, pp. 18-20.

Mnatsakanian, M. & Apostol, T. 2012 *New Horizons in Geometry*, Mathematical Association of America, pp. 216 – 217: Twisted cylinders. The authors cite more reasons why we should study 'twisted cylinder' sections, and sections of the cone only as a special case. In 500 equally revolutionary pages they write the book Archimedes would have written had he lived 2,000 years. (In case you think that's a lot of text, there are 1,000 colour illustrations. I guarantee you'll never have seen a book on elementary geometry anything like this one.)

Bostock, L., Chandler, S. & Rourke, C. 1982 *Further Pure Mathematics*, Stanley Thornes, p. 505.

### *Notes*

**1**. Have a stock of filter papers to hand so that the children don't have to rely on memory. When you fold the paper disk you produce a cone which has half the base circumference, therefore radius, of the original disk. That radius is the slant height of the cone. This gives you a '1-2-$\sqrt{3}$' triangle and a semi-apical angle of $\arcsin 1/2 = 30°$. This is a least value. For most funnels the actual figure is rather higher.
**2**. For an animation go to www.its.caltech.edu/~mamikon/TwistCyl.html .
The model shown uses Meccano parts now hard to obtain. One suggestion is a wooden equivalent using basket-weaving bases top and bottom. Shops such as Hobbycraft stock these.
Clear polycarbonate, a less brittle alternative to cast acrylic sheet ('perpsex'), gives you a plan view so that you can watch the hyperboloid close like the iris diaphragm on a camera. Approach the design technology department. They will have blanks which can be drilled but also useful advice on the whole enterprise.
There are two possible arrangements: the model shown in the text, where the string feeds through as you twist the top, and the model shown below, where fixed lengths of shearing elastic stretch. Transformations are easier to follow with the second but, because of the forces the apparatus must withstand, it requires some engineering.



Mark circular sections of the original cylinder with stickers. On both, circles stay circles and shrink. In the first, they rise and bunch. In the second, they keep their height. In the second, the figure is symmetrical with respect to the centre so you only need pay attention to the top or bottom half. Imagine the central circle fixed so that circles above rotate one way; circles below, the other. They turn in proportion to their distance from the centre. They shrink in proportion to their distance from the base (or top).
We know straight lines stay straight lines, for these are the threads. We see them lengthen and swing in towards the axis. But we can also resolve their motion into these two components: a torsional shear, which would turn them into helices; and a radial shear towards the axis.
**3.** Problems in the annual William Lowell Putnam Mathematical Competition are of the same standard as the Student Problems in The Mathematical Gazette. If you want to use them, the best place to start is [www.math.niu.edu/~rusin/problems-math/](http://www.math.niu.edu/~rusin/problems-math/) .

**4.** The locus is known as the ellipse's *orthoptic* or *director* circle. Another example of a limiting case: the *parabola*'s orthoptic circle is its directrix, a straight line therefore. When you follow the sequence of figures for section **4**, you realise it has to be.
**5.** Go to www.khanacademy.org/math/recreational-math/vi-hart/doodling-in-math-class--connecting-dots. Vi Hart is the daughter of George Hart, the moving spirit behind America's first (the world's second) hands-on maths museum, Momath.
**6.** For the solution go to www.atm.org.uk/mt238 .
**7.** The formula is stated in Proposition 1 of 'On the Measurement of a Circle', one of Archimedes' few works to survive in the original. The estimation of $\pi$ constitutes Proposition 3.
**8.** This treatment has not been traced back beyond the Renaissance. The Italian Leonardo da Vinci and Japanese Satō Moshan both used it.
**9.** This representation is due to the Jewish philosopher-scientist Rabbi Abraham bar Hiyya Hanasi, who lived in France and Spain during the 11th to 12th centuries. Go to www.cut-the-knot.org/Curriculum/Geometry/RABH.shtml .
**10.** The last result was discovered by Descartes. For the way George Polya thought he might have derived it, go to www.magicmathworks.org/Polya-Descartes. For a proof using the Euler polyhedron formula, thus reversing the historical order, go to www.magicmathworks.org/masterclasses/polyhedra-masterclass.pdf and scroll down to Part 1(d).
**11.** Go to en.wikipedia.org/wiki/Method_of_exhaustion .
**12.** I nested these polyhedra for the sake of the photograph but the children will do so uninvited. One's reminded of Kepler's model of the cosmos. (Go to en.wikipedia.org/wiki/Johannes_Kepler and scroll down to 'Mysterium Cosmographicum'.) But Kepler was following more ancient practice. I feel sure a student of comparative anthropology would recognise the model as an *archetype*. Go to en.wikipedia.org/wiki/Chinese_boxes and also consider the endless fascination for 2-3 year olds of the toy 'building beakers'.

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**Keywords**: Special/general/limiting case; Sequence;
Ellipse, hyperbola, parabola; Regular/semiregular Polygon/polyhedron.

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