Notquitohedra

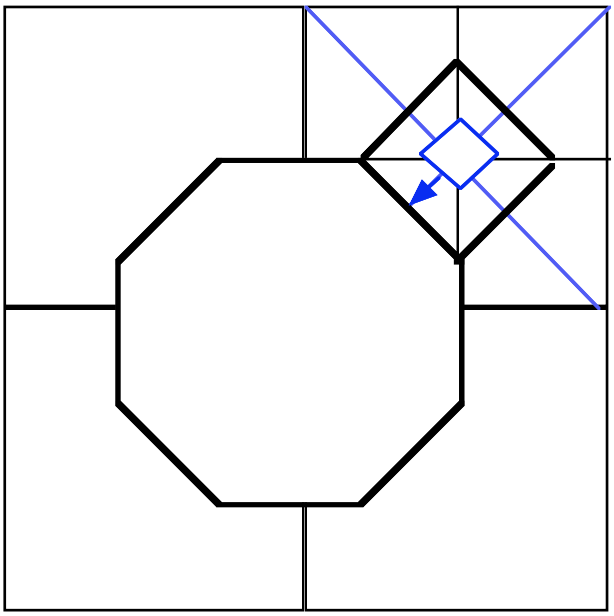
The ability of interlocking polygonal tiles like those in the Polydron Frameworks kit to flex can give rise to polyhedra whose faces don't quite fit. There was an example of a non-convex one in MT216i ('MT goes hexagonal: Impossible solid'). Here is the story of a convex family, made from three modules: the equilateral triangle, the regular decagon, and the 4-pointed star consisting of a square bordered by equilateral triangles folded in.

But we shall begin in two dimensions.

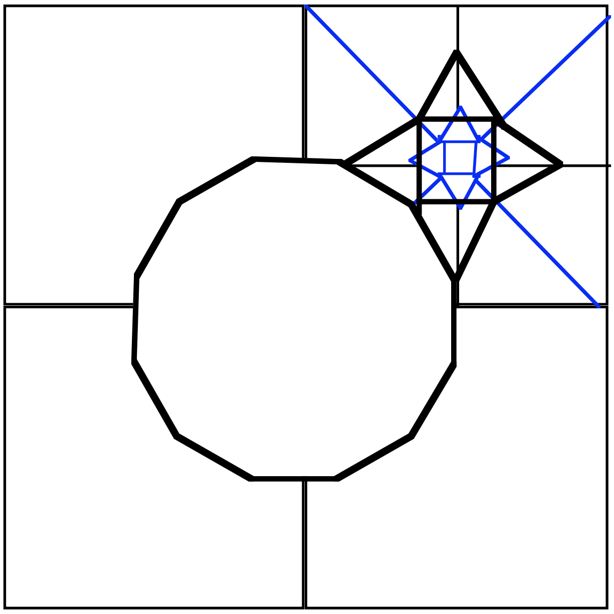
Looking at vertices, tilings have their analogues among the polyhedra. The Platonic form 6^3 (the chicken wire tiling) corresponds to 5^3 (the pentagonal dodecahedron). Truncating the vertices of 6^3 produces the Archimedean tiling 3.12^2; truncating the vertices of 5^3 produces 3.10^2.

The Platonic (squared paper) tiling 4^4 corresponds to the polyhedron 3^4 (the regular octahedron); truncating the first leads to 4.8^2; truncating the second, to 4.6^2.

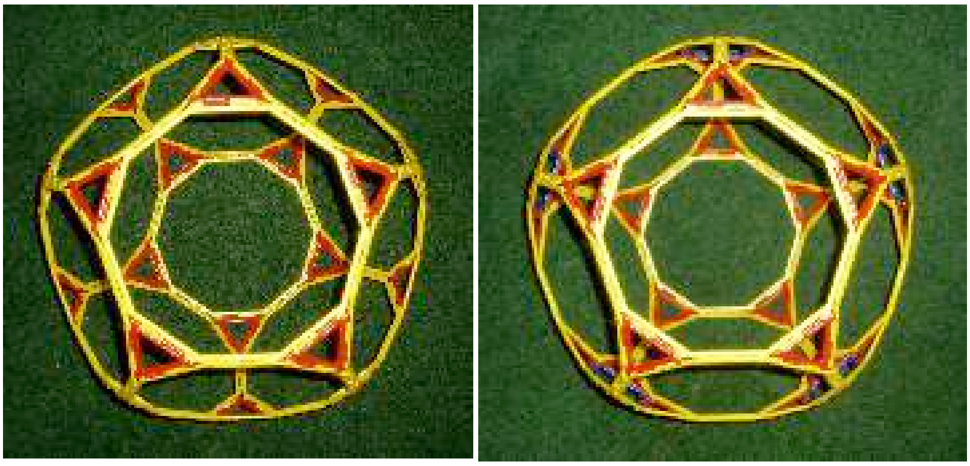
We can truncate 4^4 by making successive cuts at right angles to the square diagonals so that a small square grows until its side has the same length as the resulting octagon:



Imagine that, instead, we make cuts at $75°$ either side of the diagonal. This will leave a square bordered by equilateral triangles. That 4-pointed star will grow till we have the hybrid tiling $3.12^2 / 3.4.3.12$:



Imagine trying to build a pentagonal dodecahedron but instead of pairing the two crown-shaped half dodecahedra so that they mesh, you rotate one so that vertices coincide. Children trying to build the truncated form often do the equivalent. In this case, however, it seems to be possible to fill the interstices with (appropriately bent) 4-pointed stars. On the left is 3.10^2; on the right what appears to be a hybrid form analogous to the above tiling, call it $3.10^2 / 3.4.3.10$:



Unfortunately the dihedral angle between decagon pairs along the equator is not that between pairs away from it[1]. Since the latter correctly accommodate equilateral triangles, one concludes without further calculation that the former do not. We'll call this form the equatorial misfit. The two halves are misaligned by 1/10 of a turn.

The other day a Y6 child built what I was expecting to be that one but then I looked more closely:

It did not result from a mistake. The boy had begun by building a zone of 6 decagons so that their shared edges formed a regular hexagonal prism. As you see, the resulting solid has the symmetry of the cube. In fact you can regard the intended result as the solid of interpenetration of regular hexagonal prisms aligned along the cube's 4 space diagonals. But it wobbled slightly on the table and the decagons seemed to be symmetrically deformed like potato crisps with a 2-fold symmetry axis through their centres. Readers may like to do what I did and find a convenient dimension to check. I found that, if the skew decagons were plane, the squares would have an edge around 6% too big. A polyhedron has to obey Descartes' rule that the angle deficits of all the vertices sum to 4π. Both the last two solids satisfy that condition. In the present case, the symmetrical way in which the solid is distorted means that an excess at one vertex is compensated by a deficiency at another. The solid has the rotation axes of the cube but lacks its mirror planes. In honour of its discoverer we'll call this form the Satterthwaite solid.

This 'discovery' was no news to Paul Gailiunas, who put me in (Internet) touch with people who collect such so-called 'near-miss' polyhedra, notably Jim McNeill, who has written a program called Hedron with the following extraordinary facility: You lay out a net. Hedron folds it, tells you if it closes and the resulting distortion if it does. Go to http://www.orchidpalms.com/polyhedra/hedron.html . The software costs $25 but I can't imagine how much time Jim put into developing it.

Are there other near-miss polyhedra made from the same 3 modules?

We'll start by seeing what the Cartesian condition allows. We'll call the region including the 3 vertices of the isolated triangle, T . We'll call the region including the 8 vertices of the bent 4-pointed star, F .

The deficit at a '3.10^2' vertex is $[360-60-2(144)]° = 12°$;

the deficit at a '$3.4.3.10$' vertex is $[360-2(60)-90-144]° = 6°$.

Thus the total angle deficit t for T is $3\times 12° = 36°$;

the total angle deficit f for F is $(4\times 12)° + (4\times 6)° = 72° = 2t$.

If there are p Ts and q Fs, we have $pt+qf = 720°$, leading to $p+2q = 20$.

We note that p must be even, offering the following cases:

p	q	*name of form*
0	10	
2	9	
4	8	
6	7	
8	6	the Satterthwaite solid
10	5	the equatorial misfit
12	4	
14	3	
16	2	
18	1	
20	0	the truncated dodecahedron

Can we use further conditions to knock out some of the remaining options?

A given decagon is bordered by a Ts and b Fs. Each T accounts for 1 side, each F for 2. But there must be a free side between any such pair to accommodate a neighbouring decagon. To put that another way, the number c of neighbouring decagons must be $(a+b)$, whence:

$a+2b+c=10$,

$a+2b+(a+b)=10$,

$2a+3b=10$.

Again we note that b must be even, allowing only the values $b=0,2$. In other words a given decagon is bordered by either 5 Ts, or 2 Ts and 2 Fs, and the latter can be arranged in only two ways which are cyclically distinct: *TTFF* or *TFTF*.

Looking at the whole solid then, if there are any Fs, there must be at least 2, knocking out the case $p=18$, $q=1$.

Attempting to build $p=16$, $q=2$, we soon arrive at a decagon with an illicit border.

When q is an odd number, like 3, we have the problem that the chain of decagons with their linking Fs must close; otherwise we're left with a decagon bordered by a single F. This works for the equatorial misfit but not for the case $q=3$.

Beginning with 4 Fs, I found I was forced to insert a further 2, bringing q to 6.

I found I had built the 'equatorial misfit' of the Satterthwaite solid. Maintaining the alliteration, we'll call it the Stephenson solid. The equatorial misfit has the symmetry of the pentagonal prism; the Stephenson solid, that of the triangular prism. The two halves are misaligned by 1/6 of a turn. Note the single zone of 6 decagons parallel to the 3-axis:

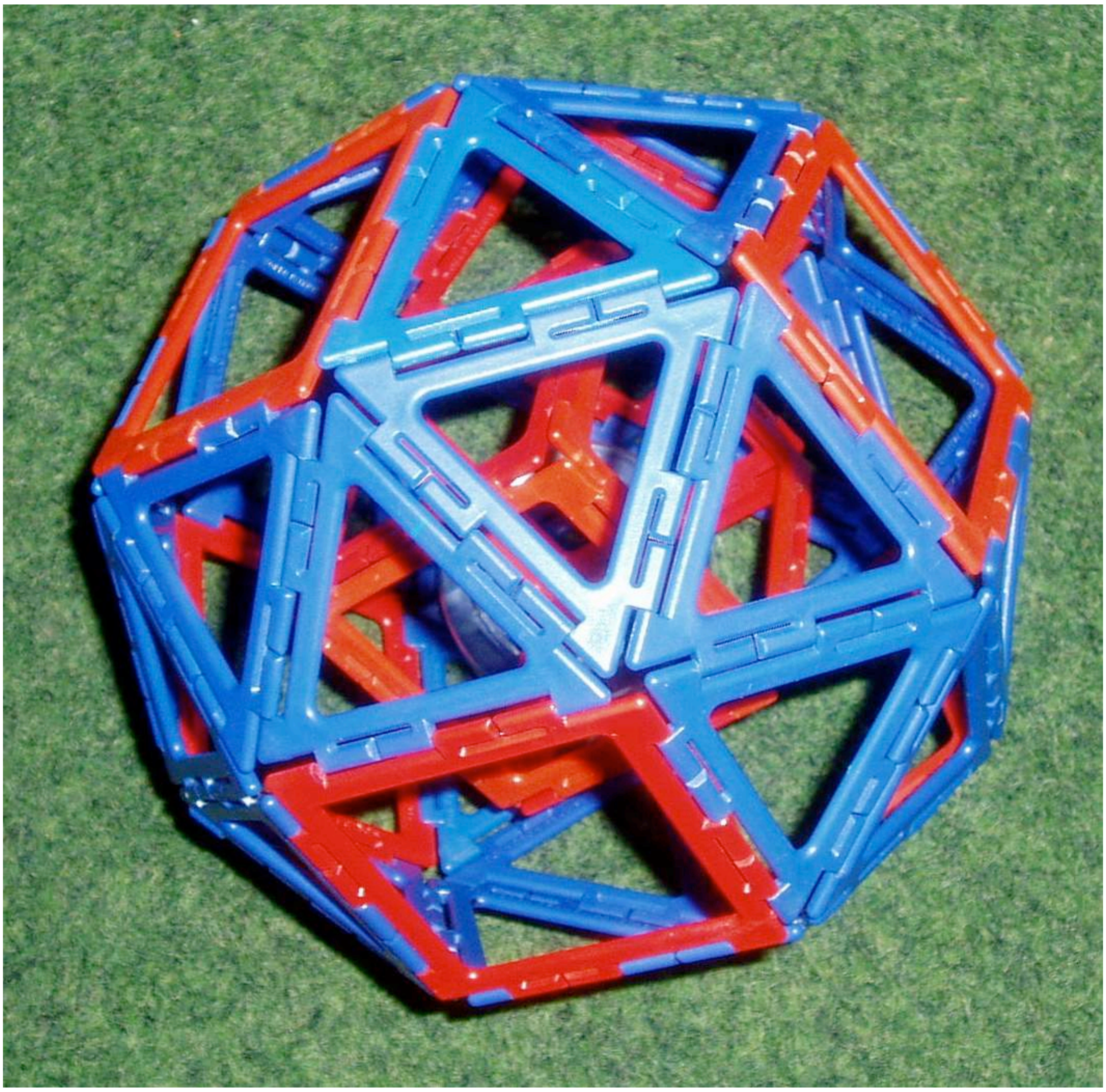
When q reaches 6, every decagon is bordered by 2 Fs. We have reached saturation: no further Fs can be added.

Have we missed any alternative forms for $q = 5,6$?

I suspect that there are only two significant forms all told: the truncated dodecahedron (and its misfit) and the Satterthwaite solid (and *its* misfit), only the very first being a bona fide polyhedron.

I can't let you go without showing you my favourite thing about the Satterthwaite solid. Imagine the decagons have disappeared. Move all the remaining bits in towards the centre, twisting them all the correct angle in the same sense, bending the petals of the 4-pointed stars the correct amount … Click! You've got a snub cube[2]:

Now repeat the process. This time the triangles vanish and the squares move inwards and twist the right amount … Click! You've got a cube:

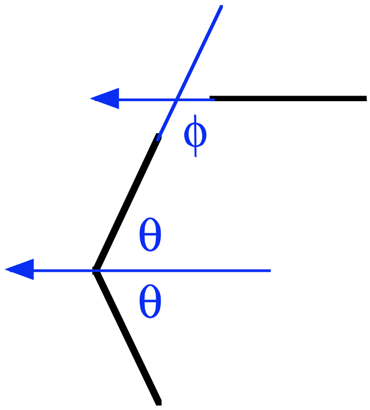


But why stop there? Remove the squares and … Oh.

Paul Stephenson is operations director of The Magic Mathworks Travelling Circus.

Note

1 The second dihedral angle is that of the original pentagonal dodecahedron. It is a good exercise for older students to work this out. They may then turn to p.87 in Cundy & Rollett to check their answer. If you draw a cross-section through our solid, you'll see that the equatorial dihedral angle is twice the supplement of this:



… and the two angles are unequal – to be equal they'd have to be 120°. Full reference:

Cundy, H.M. & Rollett, A.P. (3rd edition 1981), Tarquin

2 Cundy & Rollett, p.107