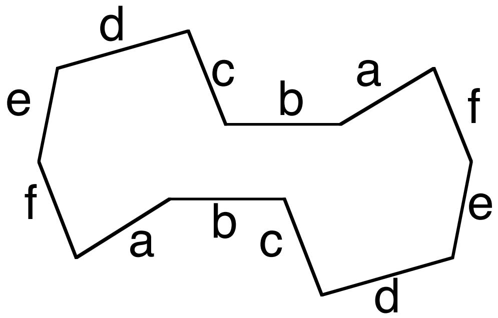
TILES FROM COVERINGS

Previn: “You’re playing *all* the *wrong notes*!”
Morecambe: “No, I am playing all the *right* notes
– but not *necessarily* in the right order.”

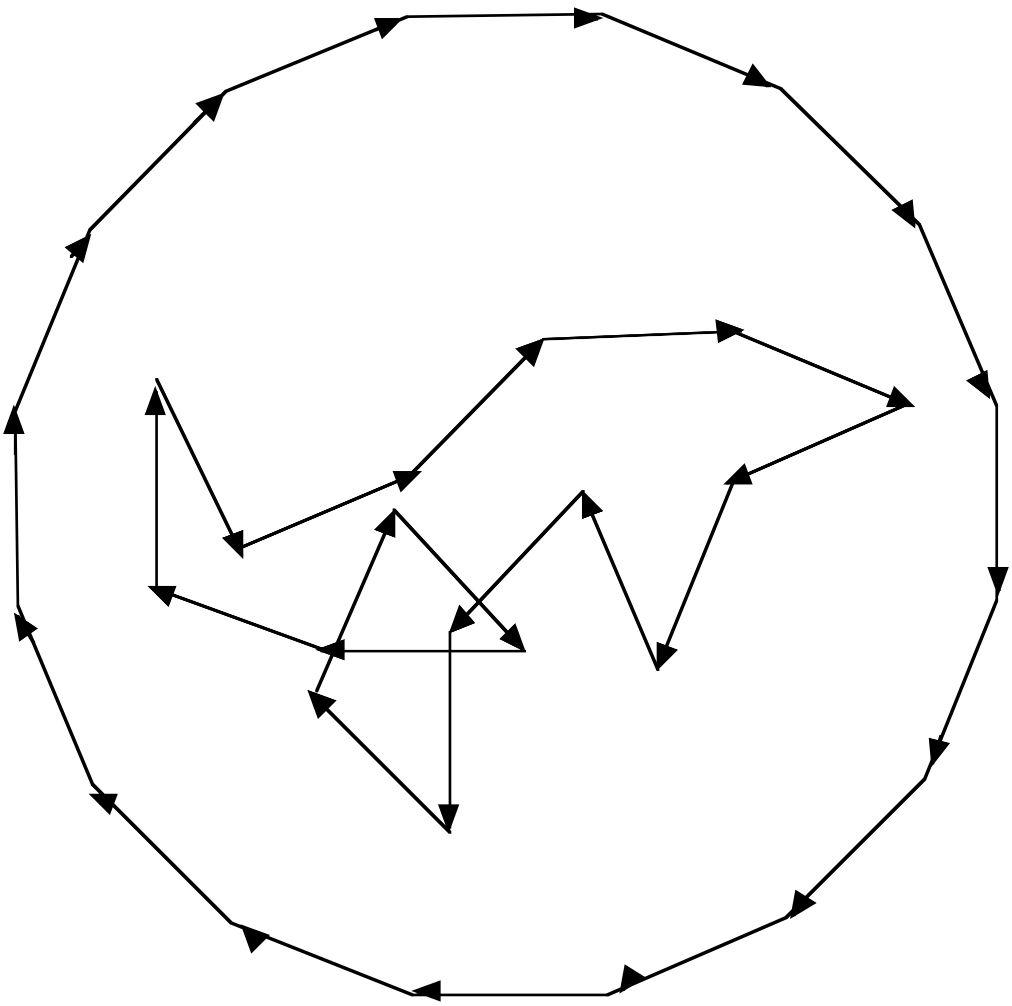
Deranged Zonagons

Throughout this article we shall be concerned with zonagons: polygons with an even number of equal sides and opposite sides parallel, i.e. polygons with equal sides so arranged that the resulting figure has half-turn symmetry. (The letters identify orientations. A repeated letter therefore indicates a parallel side.)



Specifically we shall be concerned with the most symmetrical example: the even-sided regular polygon.

If we start with a zonagon and mix up the order of sides the law of addition of vectors tells us that we shall still end up with a polygon:

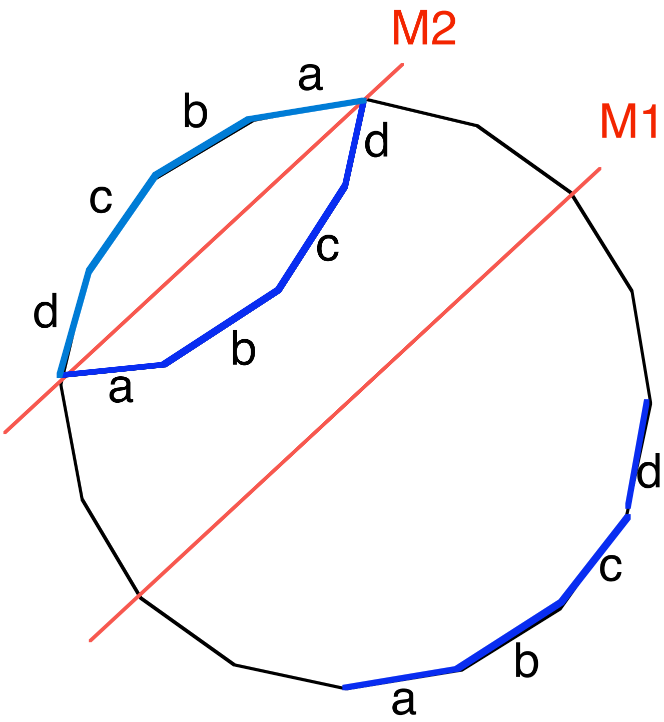


Our object in this piece is to obtain *monohedral* tilings, tilings using just one shape. The last example is not very promising. How can we improve our chances?

Some Semblance of Order: Escherification

One principle employed by Escher in his own tilings, and applied in schools throughout the land to decorate the walls of the maths department, is to take a parallelogram and complicate each pair of opposite sides in the same way, thus maintaining the translational symmetry of the tile. The activity is popular because the children make the exciting discovery that a simple mathematical idea can admit artistic complexity. We shall use it now to modify our regular $2n$-gons.

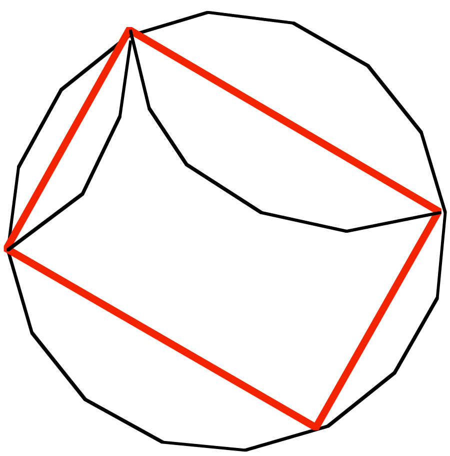
In this figure we take our regular polygon and reflect part in a chord:

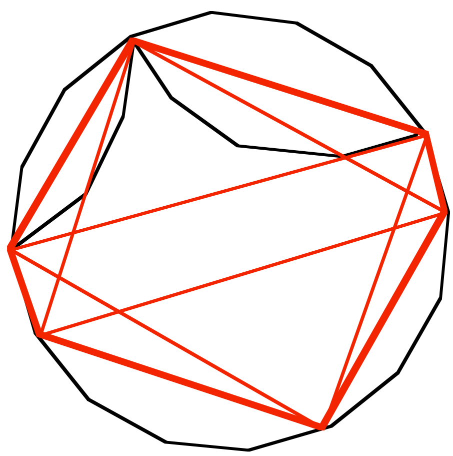


The children should prove that the boat-shaped piece is a zonagon. (It has 2 symmetry axes, necessarily perpendicular, implying half-turn symmetry. By contradiction 'or otherwise' older students should justify my phrase "necessarily perpendicular".)

We see that the part where the original letter order is reversed may be translated to match a corresponding part on the other side of the polygon. This property is unique to even-sided regular polygons. It arises from the fact that they have $2n$ symmetry axes and that the product of reflections in two parallel mirrors ($M1$, $M2$) is a translation.

If we flip adjacent sections of p and q sides, $p+q=n$, we shall have performed the Escher trick based on a rectangle. We can subdivide a section: $p+(q_1+q_2)=n$. The resulting Escher figure is a hexagon. It has opposite sides parallel but not necessarily equal, (a more general zonagon if you like). Note how Escher rectangles persist as figures inscribed in the original polygon:



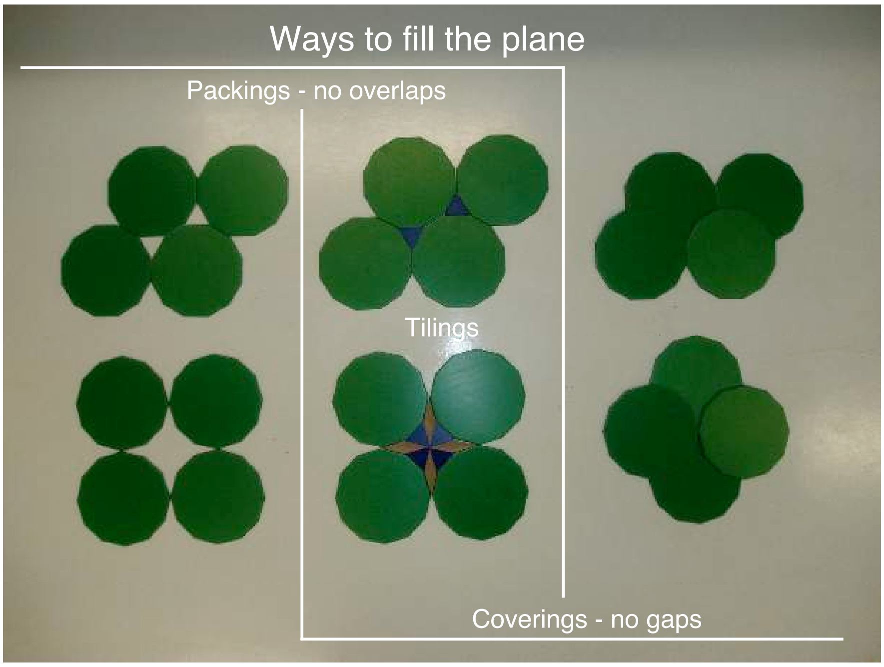


The children should convince themselves – better still, their peers - that this hexagon tiles the plane.

Escher Tiles from Coverings

This brings us to *coverings*. (We shall see why in a moment.)

Grünbaum and Shephard (1987) open their monumental work with a definition which this Venn diagram expresses:



From this, *tilings* emerge as the Goldilocks plane-filling: no gaps, no overlaps, just right.

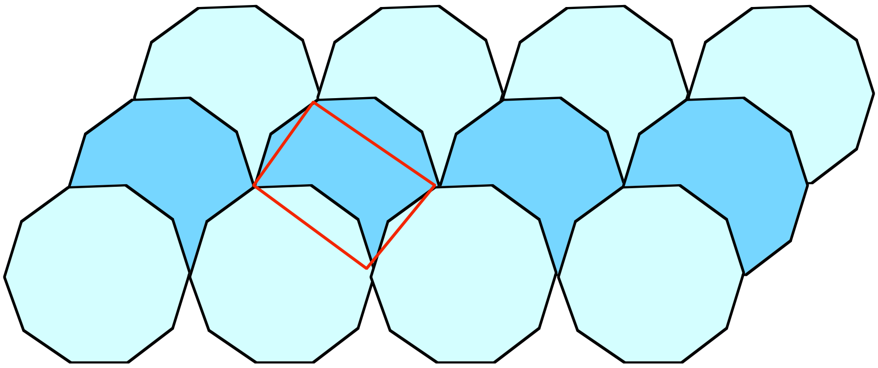
The tiles on a roof constitute a covering. But if there were no shadows to reveal that the slates overlapped you could think you were looking at a tiling. Roof tiles (usually) form a *periodic* tiling, one that can be generated by translating a part. My observation may be expressed in the theorem, offered without proof: periodic coverings imply periodic tilings.

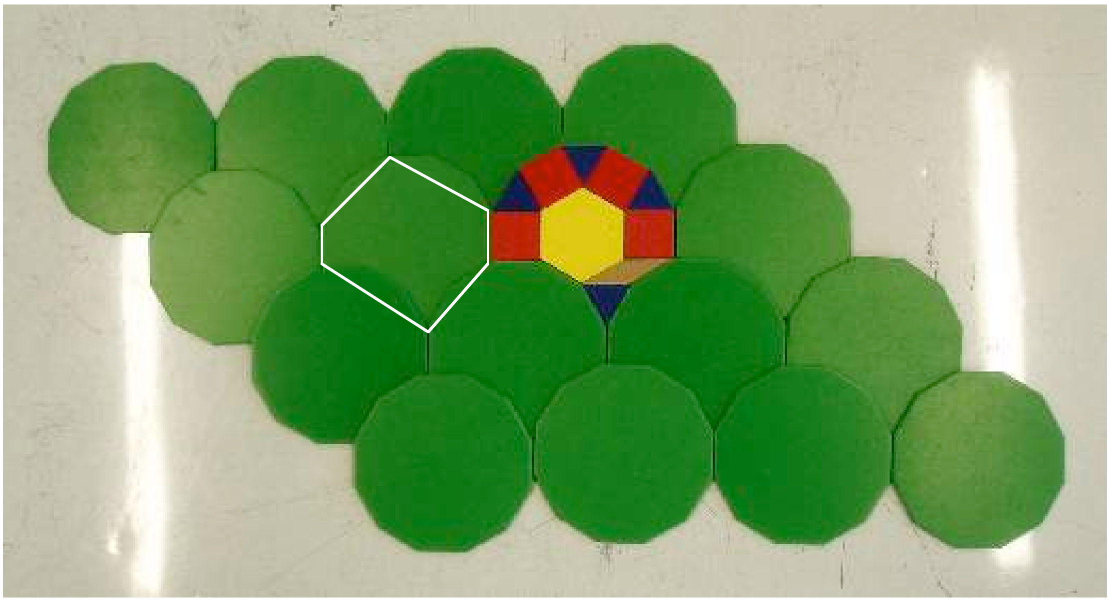
Our interest in coverings is that we can take the bite out of one side of our polygon by lapping a congruent one.

Lapping and Plaiting

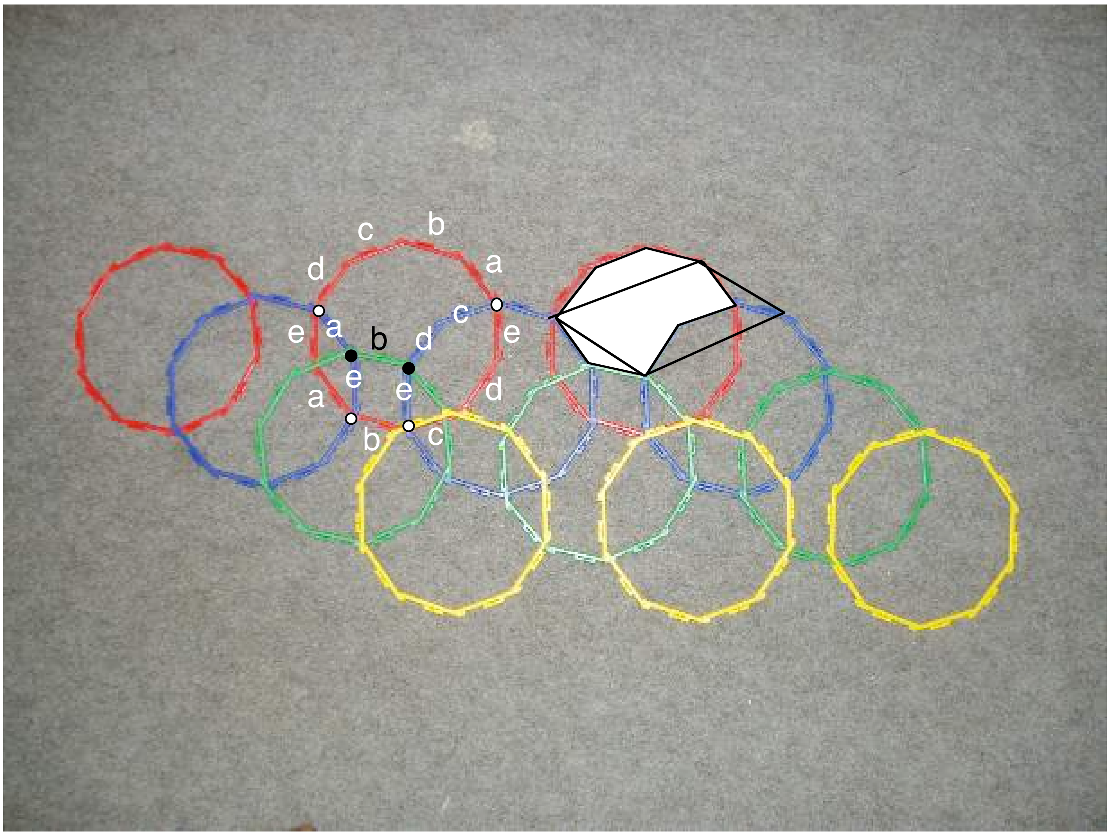
As a mathematical roofer I can either lap my slates or plait them. (If you find slate-plaiting a demanding craft, remember I'm a mathematical roofer, anxious only to maintain the distinction between a slate and a tile.) In the first arrangement I can lift off slates in a certain order without disturbing any others (though, if the roof is infinite in extent, I won't know where to start). In the second, I can lift no slate without disturbing others. (Again, that applies to an infinite expanse or the interior of a finite one.)

These two lappings illustrate the two cases mentioned earlier. The first uses 10-gons and is based on the rectangle; the second uses 12-gons and is based on the hexagon. In the second case we further dissect a tile using ‘pattern blocks’ to show the artistic possibilities in these irregular polygons.





In both those cases a bump comes opposite a bite. But this need not be so. The way we’ve lapped 10-gons in the next example is an interesting case.



10-gons are used, but the resulting tile is an 8-gon. The parallelogram used for the Escher translation is not a rectangle. Marked in white are the vertices which our lapping scheme requires to coincide. Marked in white too are the face orientations which result.

Lay out Polydron Framework 10-gons on the floor in the above way and ask the children to show that vertices at the black points coincide and the side between them has the orientation shown. They should reproduce the figure using LOGO or Geometer's Sketchpad, a useful exercise in itself.

Let us see what has happened to our original 10-gon as a permutation of the sequence of side orientations:

$$\begin{pmatrix} abcdeabcde \\ abcdaebedc \end{pmatrix}$$

We then lose the two e s. We could have started with the zonagon formed from flipping the segment $abcd$ in our original 10-gon, then transposing c and d:

$$\begin{pmatrix} abcdabcd \\ abcdabdc \end{pmatrix}$$

Because we are dealing with single tiles made from regular polygons ***the sequence of sides defines the tiling completely***. Take, for example:

$$abcdgfehidcbaefgih$$

How many sides has a tile? 18.
Where do the vertices fall? Certainly where there is a break in the letter sequence read cyclically. We find 6 here so we know we have in fact found them all:

$$abcd\uparrow gfe\uparrow hi\uparrow dcba\uparrow efg\uparrow ih\uparrow$$

We can also work out the orientations of the sides concurrent at a vertex with the two either side of the arrow. Taken in pairs, the sequences eh ,ae ,ha share a letter but reveal a third:

$$\begin{matrix} a \\ e\uparrow h \end{matrix} \qquad \begin{matrix} h \\ a\uparrow e \end{matrix} \qquad \begin{matrix} e \\ h\uparrow a \end{matrix}$$

Likewise for dg ,id ,gi . Notice too that aeh and dgi vertices alternate around a tile.

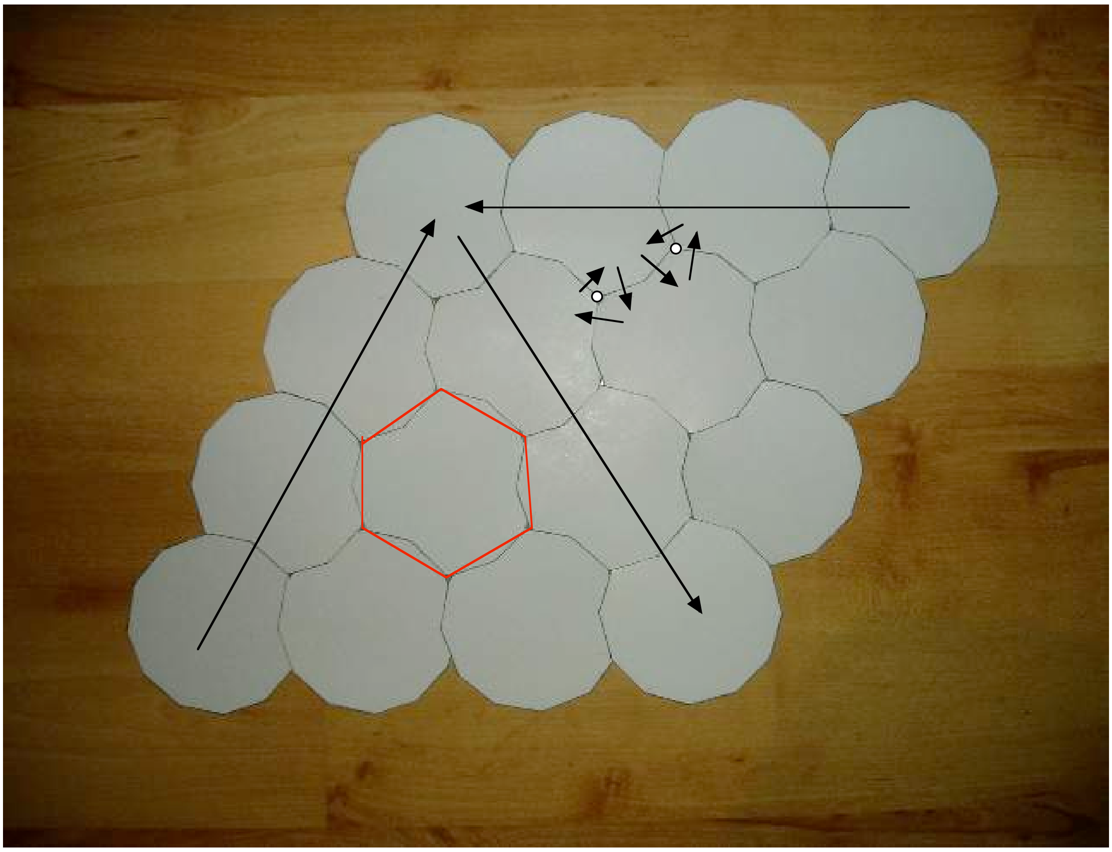
Readers may like to use LOGO or Geometer's Sketchpad to draw part of this tiling.

How about: $abcdefdcbafe$?

How many sides has a tile? 12.
Where do the vertices fall? Certainly at the 3 points marked by the solid arrows but we know there must be 4 or 6. Since stretches of corresponding length must occupy corresponding positions in the first 6 and the last 6 letters, we can locate a 4th at the position shown by the double-headed arrow:

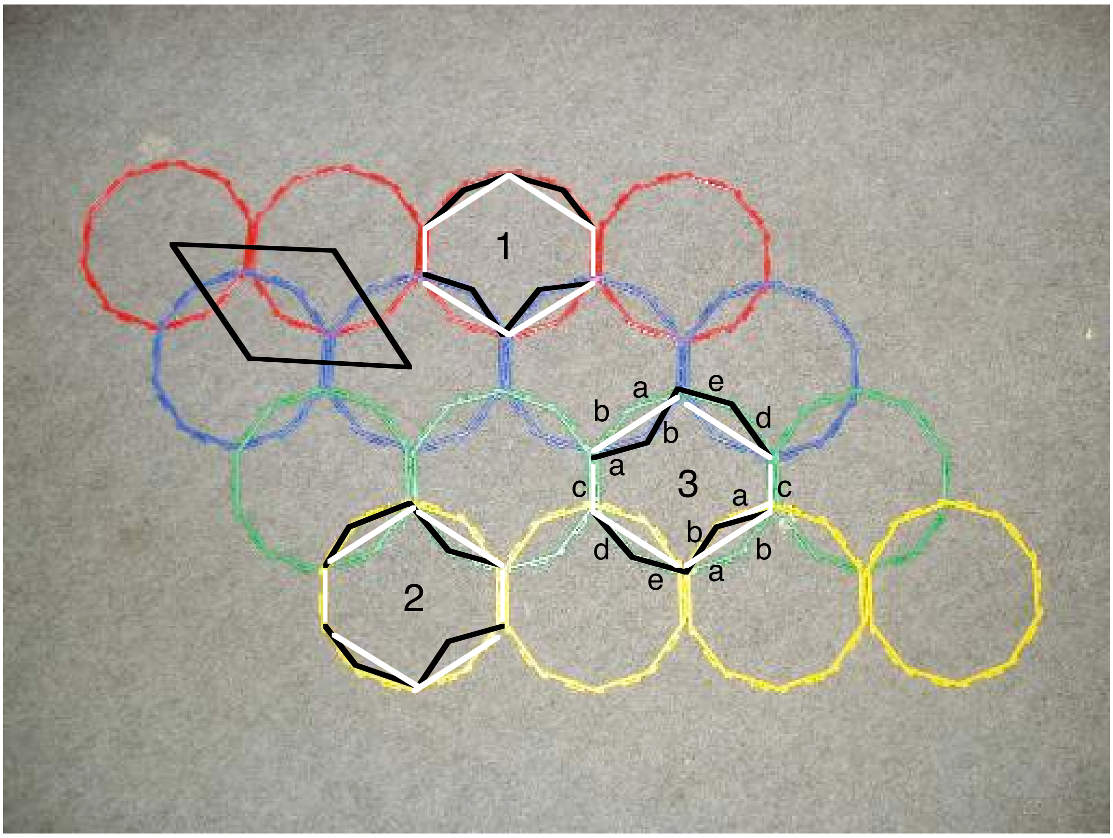
$$abcd\upuparrows ef\uparrow dcba\uparrow fe\uparrow$$

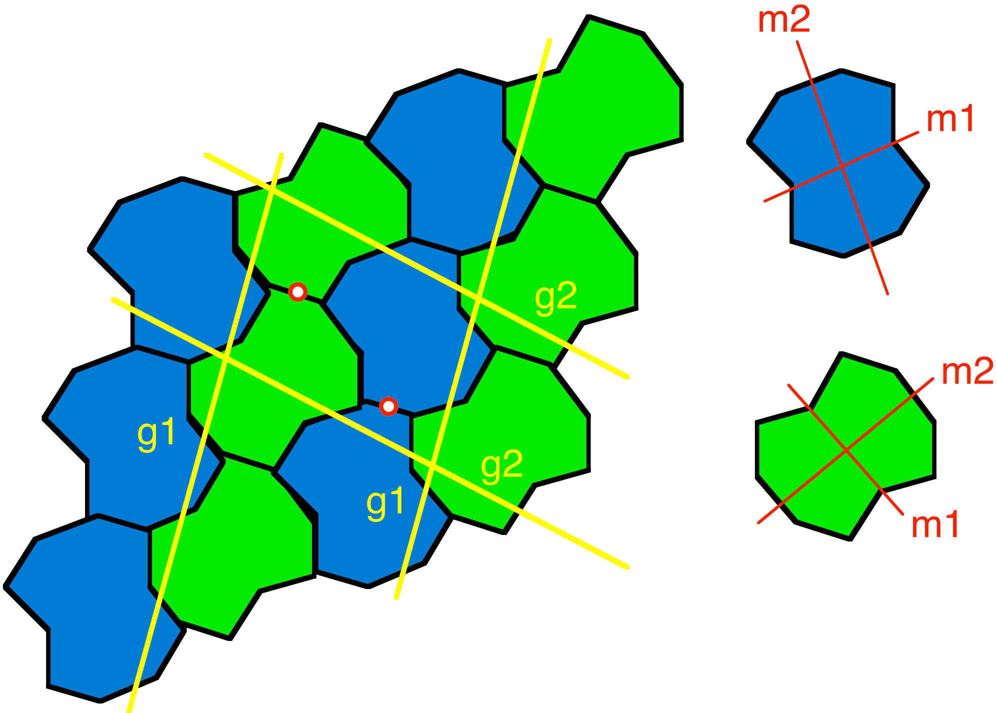
A word about lapping and plaiting. Two slates can only butt or lap and we can trace lines of slates where a left laps a right or vice versa. Plaiting describes what happens in two dimensions. Here we plait regular 12-gons to produce a familiar tiling. A vertex is a point where 3 or more slates (or tiles) meet. Notice that the sense of superposition reverses when we move from one vertex to the next:



Notice too that the symmetry of the regular 12-gon allows translations symmetrically in 3 directions based on the regular hexagon shown.

So far all our tiles have faced the same way. In the next tiling we plait 10-gons to produce a design with tiles in two orientations. Note however that the arrangement of slates without regard to lapping or plaiting allows different tiles. In the next picture we use Polydron Framework 10-gons to show this neutral starting point, defined by the parallelogram unit cell. Three different tiles are marked. Each necessarily has the area of the unit cell. **1** and **2** both have bumps corresponding to bites, result from lapping, and produce tilings with tiles in the same orientation. (Pairs of sides in which the slates simply butt against each other contribute the single edges to the Escher hexagons, shown in white.) But we have to plait the slates to produce **3**, in which a bite comes opposite a bite, and a bump opposite a bump. The two orientations are colour-coded in the diagram beneath.



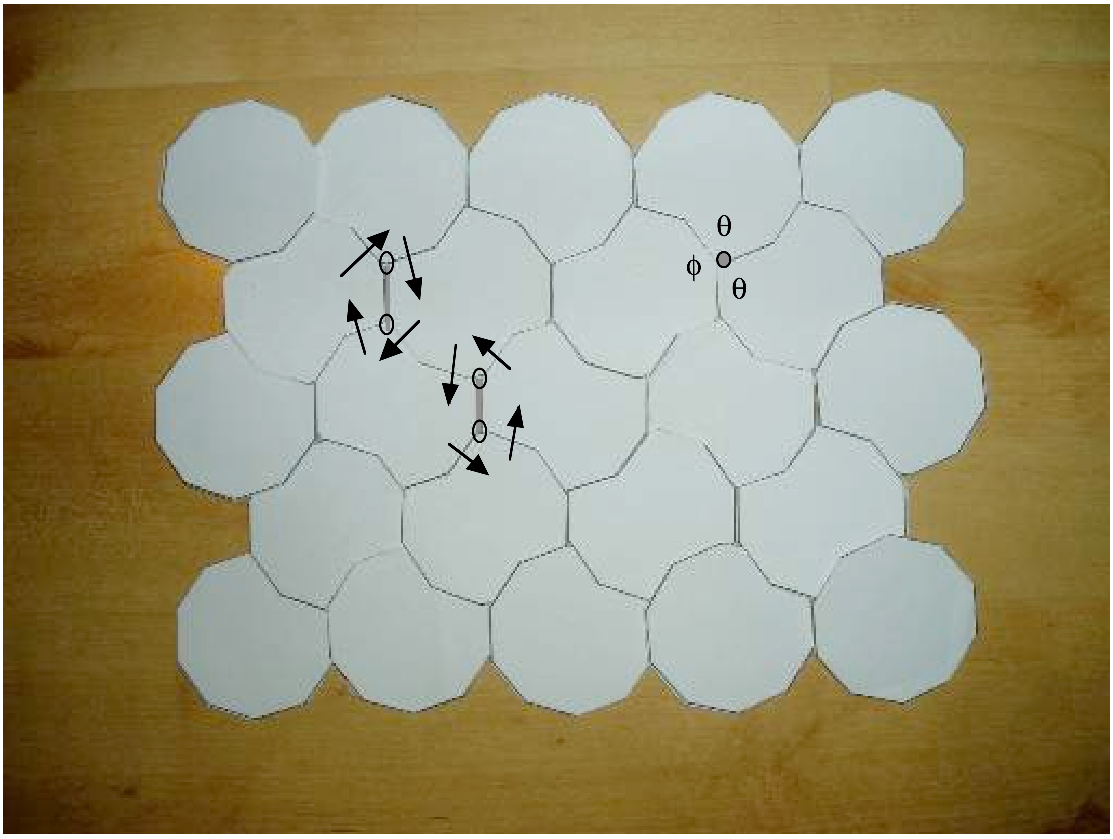


The tiles are zonagons: the permutation maintains a repeated sequence of (5) symbols:

$$\begin{pmatrix} abcdeabcde \\ bacdebacde \end{pmatrix}$$

They have two symmetry axes ($m1$, $m2$). They fit together to give a tiling with axes of glide reflection running in 2 directions ($g1$, $g2$). Notice how those box a translation cell. The tiling has centres of 2-fold rotation symmetry sited at the mid-points of sides where tiles in the same orientation abut.

The next picture shows how the slates are plaited. To see the alternating pattern shown by the 12-gon plaiting above we have to fuse vertices joined by butted edges.



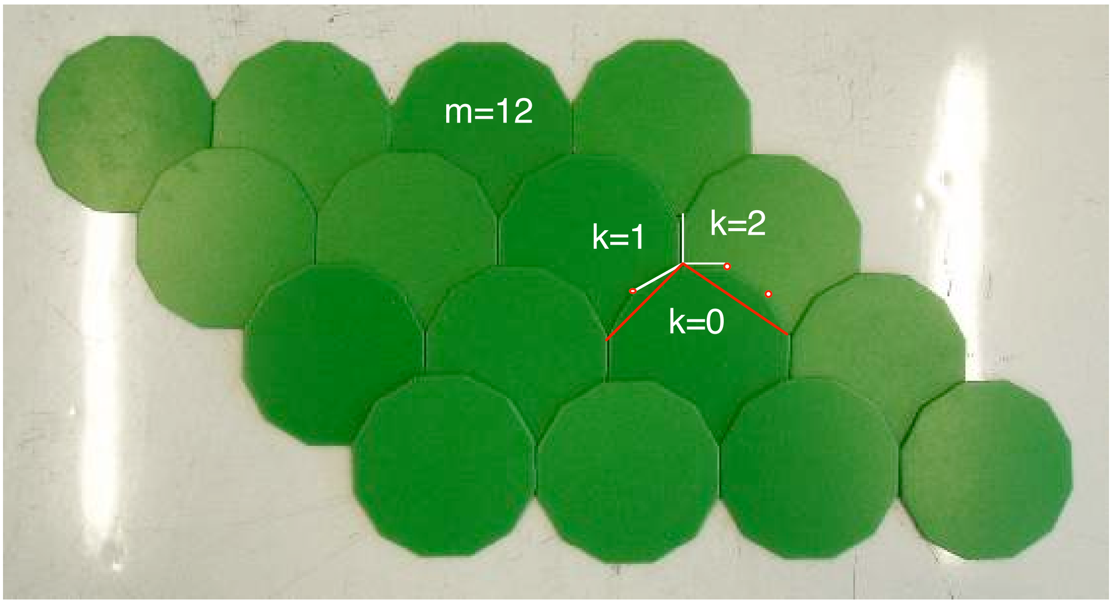
We have marked 3 angles around a particular vertex. ϕ is the interior angle of the regular 10-gon. θ is a lesser angle resulting from the flipped vertex.

So that they may apply the result elsewhere, older children may seek an expression for θ where k vertices of a regular m-gon have been flipped.

Readers may like to confirm that $\theta = \dfrac{[m-2(k+1)]\pi}{m}$. Reassuringly, the substitution $k=0$ gives the correct expression for ϕ. In the present case $m=10$, $k=1$ and we have $\theta = 108°$. Check: θ also $\dfrac{2\pi - \phi}{2} = \dfrac{360° - 144°}{2} = 108°$.

For our polygons $m = 2n$ and the identity simplifies to: $\theta = \dfrac{(n-k-1)\pi}{n}$.

Go back to the brain-shaped tile we obtained by lapping 12-gons and do the arithmetic for this vertex:



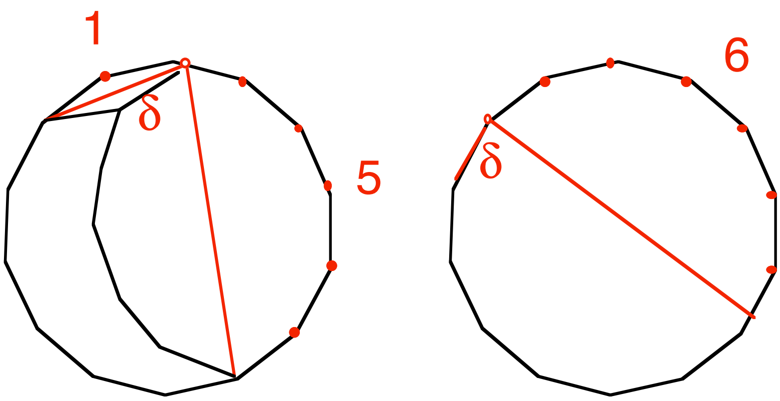
What values k_1, k_2, k_3 are possible in general at a vertex where 3 tiles/slates meet?

Readers may like to confirm that $k_1 + k_2 + k_3 = \dfrac{m-6}{2} = n-3$.

Lastly, consider a slate where two adjacent sections have been flipped, i.e. the chords in which the reflections occur share a vertex.

Confirm that we may use the same equation, summing the numbers of flipped vertices either side of the common vertex.

A nice case occurs when $\theta = 0$ so that two original slate edges come together. Our formula tells us that $k = n-1$. But we can see that straight away by folding a slate in half. Here $n = 7$, so $k = 6$. In the latter, one chord becomes a side of the original polygon, providing a trivial reflection axis, drawing our attention to the fact that δ is half the interior angle, $\phi/2$, (though we could have inferred that fact from the left-hand figure too):



You can imagine an animated sequence beginning with the right-hand figure, which we can call (0,6), and proceeding (1,5), (2,4), (3,3), (4,2), (5,1), (6,0).

As the number of sides increases, the polygon approaches a circle and δ approaches half of $180° = 90°$, as required by the 'angle in a semicircle' theorem. That in turn is a special case of the 'same segment' theorem. Since $\delta = \dfrac{\theta + \phi}{2}$, ϕ is a constant of the polygon, θ is a function only of k, and k is constant throughout a given segment, we have satisfied that too.

Screen or Floor?

One would like the 'or' to be inclusive. One would like the pupils to start with manipulatives then migrate to the keyboard where changes can be rung without limit. But that is a counsel of perfection. This whole exercise is 2-dimensional and that may swing it for you. When I was laying out my Polydron decagons on the floor the children around me watched in incomprehension as they put their own pieces to the purpose for which they were intended: building polyhedra.

Readers who nevertheless wish to know where to get the materials mentioned in the text should e-mail me. Note that there are interactive computer environments for both regular polygons and the particular shapes making up the set of 'pattern blocks'.

References

Grünbaum, B. and Shephard, G. C. 1987, *Tilings and Patterns*, W. H. Freeman and Company, ch.1, p.16.

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